REAL-TIME SIMULATION OF DYNAMICALLY DEFORMABLE FINITE ELEMENT MODELS USING MODAL ANALYSIS and SPECTRAL LANCZOS DECOMPOSITION METHODS

Cagatay Basdogan, Ph.D.

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, CA, 91109
Cagatay.Basdogan@jpl.nasa.gov

Real-time simulation of deformable objects using finite element models is a challenge in medical simulation. The finite element models developed for medical simulation applications so far can only simulate static deformations due to high computational cost. We present two efficient methods for simulating real-time behavior of a *dynamically* deformable 3D object modeled by finite element equations. The first method is based on modal analysis, which uses the most significant vibration modes of the object to compute the deformations in real-time for applied forces. The second method uses the spectral Lanczos decomposition to obtain the explicit solutions of the finite element equations. Both methods rely on modeling approximations, but lead to faster solutions than the ones obtained through direct numerical integration techniques. The finite element model of the 3D object used in the simulations was constructed from triangular surface elements interconnected to each other through nodal points. The real-time displacements of these nodal points for applied external forces were the basic unknowns of our finite element analysis. In order to analyze the deformations of the object under various loading conditions, we considered a combination of membrane and bending elements in our model. This facilitated the continuity in the formulation and enabled us to compute the displacements of nodal points for inplane and bending loads. In addition, a static condensation technique was implemented to eliminate the undesired rotational degrees of freedom and reduce the number of equations. Following the formulation of finite element equations that govern the deformable behavior of the object, we separately implemented modal analysis and Spectral Lanczos Decomposition Method (SLDM) and tested for speed and accuracy. In modal analysis, global coordinates were first transferred to modal coordinates to decouple the finite element equations. Then, modal reduction approach (selection of most significant vibration modes) was implemented to significantly reduce the number of computations. During the real-time interactions, the modal solutions obtained through Newmark numerical integration scheme were transferred back to the global coordinate frame to compute and graphically display the global deformations. We also independently applied the SLDM to the finite element equations to achieve unconditionally stable solutions. This method enables the efficient computation of explicit solutions in time and frequency domains with less effort. The SLDM was used in conjunction with the "superposition" technique to compute the global deformations of the object in real-time. We pre-computed the displacements of nodes for a unit force applied to each node along X, Y, Z directions using the explicit solutions obtained through the SLDM. We then superimposed the pre-computed solutions to estimate and graphically display the deformations of the object for the applied forces. In both methods, the errors introduced through modeling and numerical approximations were insignificant compare to the computational advantage gained for achieving real-time update rates.

1. Physically-based modeling of deformable objects for medical simulation

Modeling and simulation of soft tissue behavior in real-time is a challenging problem. Once the collision point is identified, the problem centers around tool-tissue interaction. This involves a realistic haptic force-feedback to the user and a realistic graphical display of tissue behavior depending on what surgical task (e.g. suturing, grasping, cutting, etc.) the user chooses to perform on the tissue. This is a nontrivial problem which calls for prudence in the application of mechanistic and computer graphics techniques in an endeavor to create a make-believe world that is realistic enough to mimic reality but efficient enough to be executable in real time. Soft-tissue mechanics is complicated not only due to non-linearities, rate and time dependence in material behavior, but also because the tissues are layered and non-homogeneous. Although mechanics community has developed sophisticated tissue models in the past, their integration with medical simulators has been difficult due to real-time requirements. With the addition of haptic displays, this task became even more challenging since a haptic loop typically requires much higher update rate than a visual loop.

In this paper, we discuss fast numerical techniques for real-time simulation of ***dynamically*** deformable bodies modeled using finite elements. We discuss two techniques, modal analysis (Basdogan, 1999; Basdogan et al., 2000) and spectral Lancsoz Decomposition for the real-time solution of finite element models along with the description of our own FEM. FEM techniques, though they demand more CPU time and memory, seem more promising than other modeling techniques in integrating tissue characteristics into organ models. However, the finite element models developed for surgical simulation applications so far can only simulate static deformations with limited similarities to the actual tissue behavior (Bro-Nielson and Cotin, 1996). More recently, Cotin et al. (1999) presented a model for simulating the nonlinear response of soft tissues, but only static deformations were considered. We made modeling simplifications to simulate deformable *dynamics* of virtual organs using finite elements.

2. Our Finite Element Model

<u>The finite element formulation:</u>

The 3D model of the object was approximated as an assemblage of discrete triangular surface elements interconnected to each other through nodal points. The displacements of these nodal points for applied external forces were the basic unknowns of our FEM analysis. The coordinates of vertices (nodal points), the polygon indexing, and the connectivity of vertices were derived from the geometric model of the object. In order to analyze the deformations of the object under various loading conditions, we considered a combination of membrane and bending elements in our model. This facilitated the continuity in the formulation and enabled us to compute the displacements of nodal points in X, Y, Z directions for both inplane and bending loads. For each node of the triangular element subjected to inplane loads, the displacements (u, v) in the local x and y directions were taken as the degrees of freedom. The resulting system equations were expressed in local coordinate system as

$$F_m = \left[k'_m\right] U_m \qquad \text{(Eq. 1)}$$

where, m represents the membrane action. Similarly, for each node of the triangular element subjected to bending loads, the displacement in local direction and slopes (rotations) about the local x and y axes were considered. The relation between the vertex displacements and forces were written as

$$F_b = \left[k'_b\right] U_b \qquad \text{(Eq. 3)}$$

where, b represents the bending action. In order to obtain the local stiffness matrix for each triangular element, the inplane and bending stiffness matrices were combined. Since 6 degrees of freedom were assumed for each of the vertices, the resulting combined local stiffness matrix ($[k^e]$) became 18x18 for each of the triangular elements of the object.

Transformations:
The stiffness matrix derived in the previous section utilizes a system of local coordinates. However, the geometric model of the object was generated based on the global coordinate system. In order to apply the computations described in the previous section, a transformation from the global coordinates to local coordinate system was required (Zienkiewicz, 1990). The forces and displacements of a node were transformed from the global to the local system. Following the finite element analysis, it was necessary to transform the results back to the global reference frame. The new positions of each nodal point for a given force were computed using system equations in the local coordinate system. Then, these new coordinates were transformed to the global coordinate system in order to update the graphics.

Assembly of Element Stiffness Matrices:
The element stiffness matrices (K^e) were put together to construct the overall stiffness matrix (K). This process can be symbolically written as

$$K = \sum_{e=1}^{p} K^e \quad \text{(Eq. 14)}$$

where, p represents the number of triangles.

Implementation of Boundary Conditions:
In order to obtain a unique solution for finite element equations, at least one boundary condition must be supplied. The implemented boundary conditions modify the stiffness matrix K and make it nonsingular. There are multiple ways of implementing boundary conditions as discussed in the literature (Huebner et al., 1995). The easiest way to implement the boundary conditions is to modify the diagonal elements of the K matrix and the rows of the force vector F at which the boundary conditions will be applied. In our model, one end of the object was fixed, which implied zero displacements for the associated fixed nodes. To implement this boundary condition, diagonal elements of the K matrix and the rows of the F vector associated with those fixed nodes were multiplied by a large number. This procedure makes the unmodified terms of K very small compared to the modified ones.

Eliminating the Rotational Degrees of Freedom (Condensation):
The global stiffness matrix was assembled as a symmetric square matrix and its length was six times the number of nodes of the object (recall that we defined 6-dof, 3 translations and 3 rotations, for each node of the object). Although the rotational dof were necessary for the continuity of the solution, their computation was not required for our simulations. Since our main interest was to obtain the translational displacements of each node, the overall stiffness matrix K was condensed such that the rotational dof were eliminated from the formulation. In addition, the condensation of K matrix automatically reduced the number of computations to half, which was helpful for achieving real-time rendering rates. To condense the K matrix, we first partitioned the displacement and load vectors of the static problem as

$$\begin{bmatrix} K_{tt} & K_{tr} \\ K_{rt} & Krr \end{bmatrix} \begin{bmatrix} U_t \\ U_r \end{bmatrix} = \begin{bmatrix} F_t \\ F_r \end{bmatrix} \quad \text{(Eq. 15)}$$

where, subscripts t and r represent the translational and rotational dof respectively. Then, we set the forces acting on the rotational degrees of freedom to zero, and condensed the stiffness matrix as

$$K_{condensed} = K_{tt} - K_{tr}(K_{rr})^{-1} K_{rt} \quad \text{(Eq. 16)}$$

The condensed stiffness matrix was a full square matrix and its length was three times the number of nodes of the object.

3. Modal Analysis Method

The dynamic equilibrium equations for a deformable body using FEM can be written as

$$M\ddot{U} + B\dot{U} + KU = F \quad \text{(Eq. 17)}$$

where, M and B represent the mass and the damping matrices respectively. Once the equations of motion for deformable body are derived, the solution is typically obtained using numerical techniques. However, as it will be discussed in detail later, the real-time display of FEM becomes increasingly more difficult as the number of elements is increased. A particular choice of the mass and damping matrices significantly reduces the number of computations. If the mass matrix is assumed to be diagonal (mass is concentrated at the nodes) and the damping matrix is assumed to be linearly proportional with the mass matrix ($B = \alpha M$), the equations are simplified. A further modeling simplification can be implemented if we assume that high frequency vibration modes contribute very little to the computation of deformations and forces. If dynamic equilibrium equations are transformed into a more effective form, known as modal analysis, real-time solutions can be obtained with very reasonable accuracy. Pentland and Williams (1989) demonstrated the implementation of this technique in graphical animation of 3D deformable objects. In modal analysis, global coordinates are transferred to modal coordinates to decouple the differential equations. Then, one can either obtain the explicit solution for each decoupled equation as a function of time or integrate the set of decoupled equations in time to obtain the displacements and forces. Moreover, we can also reduce the dimension of the system, as well as the number of computations, by picking the most significant vibration modes and re-arranging the mass, damping, and stiffness matrices. This procedure is also known as modal reduction.

We implemented the modal reduction approach to achieve a real-time performance. The errors introduced by the modal reduction were insignificant compare to the computational advantage gained through the approximation.

3.1. Modal Transformation:

We defined the following transformation to transform our differential system into a modal system:

$$U(t)_{nx1} = \Phi_{nxn} X(t)_{nx1} \quad \text{(Eq. 18)}$$

where, Φ is the modal matrix, U and X represent the original and modal coordinates respectively. The modal matrix was obtained by solving the eigen problem for free undamped equilibrium equations:

$$K\phi = \omega^2 M\phi \quad \text{(Eq. 19)}$$

where, ω and ϕ represent the eigenvalues (i.e. vibration frequencies) and eigenvectors (i.e. mode shapes) of the matrix ($M^{-1}K$) respectively. The modal matrix was constructed by first sorting the frequencies in ascending order and then placing the corresponding eigenvectors into the modal matrix in column-wise format ($0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \ldots \leq \omega_n$, $\Phi = [\phi_1, \phi_2, \phi_3, \ldots, \phi_n]$).

Finally, a set of decoupled differential equations (i.e. modal system) was obtained using the modal matrix and the transformation defined by Eq. 18:

$$\ddot{X}_i + \alpha_i \dot{X}_i + \omega_i^2 X = f_i \qquad i = 1, \ldots, n \qquad \text{(Eq. 20)}$$

where, n is the degrees of freedom (dof) of the system, $\alpha_i = 2\omega_i \zeta_i$, and $f_i = \phi_i^T F$ are the modal damping and force respectively. Note that ζ is known as the damping ratio or modal damping factor.

3.2. Modal Reduction:

Once the equations for modal system is derived, the explicit solution can be obtained using the Duhamel integral (see Bathe 1996 and Shabana 1996). Alternatively, one can use numerical integration techniques to obtain the modal solution. However, we can, at this stage, implement modal reduction approach to significantly reduce the number of computations. For a deformable body under external loading, the high frequency modes do not significantly contribute to the displacements. Hence, the final, deformed, shape of the object can be approximated by "r" number of low frequency modes (i.e. the first "r" columns of the modal matrix). To implement the modal reduction, we first reduced the modal matrix by picking only a few significant modes (i.e. the first "r" columns of the modal matrix). Our differential system for modal coordinates was reduced to "r" number of equations, which were then solved using a numerical integration technique:

$$\ddot{X}^R_i + \alpha \dot{X}^R_i + \omega_i^2 X^R = f^R_i \qquad i = 1, \ldots, r \qquad \text{(Eq. 21)}$$

where, the superscript R represents the reduced system. We then transferred the modal coordinates back to the original coordinates using the following transformation:

$$U(t)_{nx1} = \Phi^R{}_{nxr} X^R(t)_{rx1} \qquad \text{(Eq. 22)}$$

3.3. Numerical Integration:

Numerical integration techniques are typically used to solve the differential equations that arise from FEM. Various integration schemes based on finite difference equations have been suggested in the literature for the dynamic analysis of FEM (see Bittnar and Sejnoha, 1996; Bathe 1996). In our case, real-time performance and the stability of solutions for various loading, initial, and boundary conditions are both equally important. For example, the central difference method appears to be fast and simple to implement, the solutions becomes unstable if the integration step (Δt) is greater than (T_n / π), where T_n is the shortest period of vibration. Bathe (1996) suggests Newmark numerical integration procedure due to its optimum stability and accuracy characteristics. The Newmark method is implicit and also known as the "average acceleration" method.

Using the Newmark method, we first formulated the displacement and velocity of each reduced modal coordinate at $t + \Delta t$ as

$$ {}^{t+\Delta t}\dot{X}^R = {}^t\dot{X}^R + [(1-\delta)\,{}^t\ddot{X}^R + \delta\ {}^{t+\Delta t}\ddot{X}^R]\Delta t \qquad \text{(Eq. 23)} $$

$$ {}^{t+\Delta t}X^R = {}^tX^R + {}^t\dot{X}^R\ \Delta t + [(\frac{1}{2}-\eta)\,{}^t\ddot{X}^R + \eta\ {}^{t+\Delta t}\ddot{X}^R]\Delta t^2 \qquad \text{(Eq. 24)} $$

where, η and δ are parameters that can be determined to obtain integration accuracy and stability (solutions become unconditionally stable for $\eta = 1/4$ and $\delta = 1/2$). Then, the equilibrium equation for each reduced modal coordinate was formulated at $t+\Delta t$ as

$$ {}^{t+\Delta t}\ddot{X}^R + \alpha\ {}^{t+\Delta t}\dot{X}^R + \omega^2\ {}^{t+\Delta t}X^R = {}^{t+\Delta t}f^R \qquad \text{(Eq. 25)} $$

Finally, we substituted the displacement and velocity formulations into the equilibrium equation derived for $t+\Delta t$ and obtained a system that looks quite similar to the static analysis (Please note that we assume the mass matrix is diagonal and the damping matrix is linearly proportional with the mass matrix such that $B = \alpha M$):

$$ \hat{F}\hat{U} = \hat{K} \qquad \text{(Eq. 26)} $$

where, $\hat{F}, \hat{U}, \hat{K}$ are modified force and displacement vectors and modified stiffness matrix.

4. The Spectral Lanczos Decomposition Method:

Druskin and Knizhnerman (1994) have recently introduced a new technique called Spectral Lanczos Decomposition method (SLDM), which can explicitly solve the Maxwell's diffusion equations used in electromagnetic theory for multiple frequencies with minor additional computational cost. Zunoubi (1998) tested the efficiency and validity of this technique by studying the resonant frequencies of various microwave cavities. We have adopted this technique into our studies to solve the finite element equations in real-time for simulating the behavior of deformable objects.

In order to solve the finite element equations using the SLDM, we first rearrange the terms of the dynamical equations as:

$$ \left[\frac{\partial}{\partial t^2}I + \alpha\frac{\partial}{\partial t}I + K'\right]E' = F' $$

where, $K' = M^{-1/2}KM^{-1/2}$, $E' = M^{1/2}U$, $F' = M^{-1/2}F$. If we transfer the equations to Laplace domain and assume the applied force is constant with a magnitude of F_o, we obtain

$$ E'(s) = \frac{F_o}{s(s^2I + \alpha s + K')} $$

Using separation of variables:

$$ E'(s) = \frac{A}{s} + \frac{Bs + C}{(s^2I + \alpha s + K')} $$

where, $A = F_o / K'$, $B = -F_o / K'$ and $C = -(\alpha F_o)/K'$. If we apply the inverse Laplace transform, we obtain the time-domain solutions:

$$E'(t) = F_o \frac{1}{K'}(1 - e^{\frac{-\alpha t}{2}} Cos(\sqrt{K' - (\alpha^2/4)I}\, t) + \frac{\alpha}{2}\frac{1}{\sqrt{K' - (\alpha^2/4)I}} e^{\frac{-\alpha t}{2}} Sin(\sqrt{K' - (\alpha^2/4)I}\, t)$$

Now, if we can approximate the K' matrix as a diagonal matrix, we can easily obtain the time domain solutions. To achieve our goal, we implement the Lanczos scheme (for more information about Lanczos scheme, see Datta and Trefethen, L., Bau) with complete reorthogonalization using Householder transformations. The idea of using Householder matrices to enforce the orthogonality appears in Golub (1996). We first compute the tridiagonal Ritz approximation (T) of the matrix K':

$$Q^T K' Q = T$$

where, $Q = [q_1, q_2, \ldots, q_M]$ is an orthogonal matrix (The vectors $q_1, q_2, \ldots, q_M$ are called Lancsoz vectors) M is the size of the square K' matrix (also the number of equations) and T is the tridiagonal matrix which is determined using the complete reorthogonalization Lancsoz scheme that relies on Householder transformations (see Appendix). Then, if we define the Λ and V are the eigenvalues and eigenvectors of the matrix T, one can write matrix T as:

$$T = V \Lambda V^T$$

where, $\Lambda = diag[\lambda_1, \lambda_2, \cdots, \lambda_M]$.

Finally, $E'(t)$ can be approximated as:

$$E'(t) = F_o Q V \left[\frac{1}{\Lambda}(1 - e^{\frac{-\alpha t}{2}} Cos(\sqrt{\Lambda - (\alpha^2/4)}\, t) + \frac{\alpha}{2}\frac{1}{\sqrt{\Lambda - (\alpha^2/4)}} e^{\frac{-\alpha t}{2}} Sin(\sqrt{\Lambda - (\alpha^2/4)}\, t) \right] V^T e_1$$

where, $e_1 = (1,0,0,\cdots,0)^T$ is a unit vector. If $\Lambda_i - (\alpha^2/4) < 0$, then, Cos and Sin functions in the i^{th} equation should be replaced by $Cosh$ and $Sinh$ functions.

4.1. Superposition

Following the explicit solution of finite element equations, we generate, *an "impedance map"* of the entire 3d object. This will involve pre-computations of realistic displacement fields by applying unit loads along each nodal degree of freedom, while assuring the positive definiteness of the structure. Such a look-up table can be pre-computed well ahead of the actual simulation process. Then, it can be used to display realistic deformation fields during real-time simulation using the superposition technique. The superposition approach calculates the response of the complete system by superimposing (i.e., adding together) the individual responses of the nodes. To calculate the response of a certain node, only the responses of neighboring nodes can be used to reduce the number of computations (i.e. define a *radius of influence* and consider the contribution of nodes which are within the radius of influence of the contacted node). The superposition approach provides a solution, which is an approximation to the exact solution. However, this approximation is usually reasonably accurate (please note that we use a linear FEM model for simulating the behavior of tissues. However, tissue behavior is highly nonlinear and this approximation would not work).

5. Discussion

In this study, a modal analysis and Spectral Lanczos Decomposition method were introduced to run our *dynamic* finite elements in real-time. Our aim was to develop fast solutions for simulating deformable dynamics of soft tissues using FEM. Although our finite element model can simulate the real-time dynamics of a deformable object, it only approximates the characteristics of living tissues with certain degree due to the stringent requirements of real-time simulation. However, we should point out that both techniques are computationally faster than the direct solutions. For example, the direct numerical integration of the original differential system results in $O(n^2)$ floating point operations. However, the solution using modal analysis leads to $O(n \log n)$ flops. Therefore, the simulation using direct integration will be increasingly more difficult as n (i.e. degrees of freedom or the number of nodes of the object) increases. While the modal approach provides numerical solutions, the SLDM can return the explicit solutions of finite element equations. The SLDM, when combined with superposition technique, can be very efficient in simulating "point-based" interactions with deformable objects. This feature is especially appealing for haptic rendering. Since most of the existing haptic devices and the rendering algorithms rely on point interactions with virtual objects to compute and reflect forces, computing the "*impedance* map" of the object enables us to estimate the deformation field easily at the contact point during real-time interactions.

Appendix:

Lancsoz With Complete Reorthogonalization Using Householder Transformations: Here, we first describe the basic Lancsoz algorithm which cannot be implemented directly due to the fact that ortogonality breaks down easily. We, then, discuss the practical implementation of Lancsoz algorithm using complete orthogonalization using Householder transformations. Now, recall the tridiagonal T matrix used in equation ($Q^T K'Q = T$). The elements of this matrix can be defined as

$$T_{i,i} = \mu_i \qquad i = 1,,2,\cdots,M$$
$$T_{i,i-1} = T_{i-1,i} = \beta_i \qquad i = 1,2,\cdots,M-1$$

so that Eq. ($Q^T K'Q = T$) can be written as

$$K'q_i = \beta_{i-1}q_{i-1} + \alpha_i q_i + \beta_i q_{i+1} \qquad i = 1,,2,\cdots,M$$

where, $\beta_0 q_0 = 0$. Now, if we multiply both sides of this equation by q_i^T to the left and observe the orthonormality condition ($q_i^T q_i = 1$ and $q_i^T q_j = 0$), we obtain $\alpha_i = q_i^T K' q_i$. Then, if we define $r_i = (K' - \alpha_i I)q_i - \beta_{i-1}q_{i-1}$, then we get $q_{i+1} = \dfrac{r_i}{\beta_i}$, where $\beta_i = \|r_i\|_2$.

It is obvious that orthogonality will be lost if β_i is a small. In fact, Golub (1996) states that the loss of orthogonality always occurs in practice, which results in deterioration in the quality of eigenvalues of T matrix. An obvious way to reduce this is to reorthogonalize each newly computed Lancsoz vector against its predecessors. The details of complete reorthogonalization scheme can be found in Golub (1996).

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